Higher Representation Theory for Excitations in Gauge Theories

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Introduction

Operators and Excitations

- Two interesting things to classify in QFT
  - **Operators**
  - **Excitations**
- $\text{Operators} \not\cong \text{States}$ in a non-conformal QFT
- Different Representation Theories
  - (Genuine) Operators $\Leftarrow$ (higher) group representation
  - Excitations $\Leftarrow$ (higher) **groupoid algebra** representation

Today's message

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- Representation theory for **excitations**
  - **Higher groupoid representation**
  - Depends on **microscopic symmetry** and **infrared phase**
  - **Systematic** classification of excitations
  - **Higher** representation $\Rightarrow$ **extended** excitations

- Meaning of "gauge charge"
- ("Non-invertible" symmetries $\Rightarrow$ higher strip algebra representation)

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Representation theory in 1+1d (Cordova, Holfester, and Ohmori 2024)

Clay Córdova

Nicholas Holfester

Symmetry Preserved Phase

- G: finite group microscopic symmetry
- A single vacuum $|0\rangle$.
- Symmetry preserves the vacuum $g|0\rangle = |0\rangle$.
- Excitations are (and thus whole Hilbert space is)in a representation:

$$g \cdot |e\rangle = \rho_e(g)|e\rangle,$$

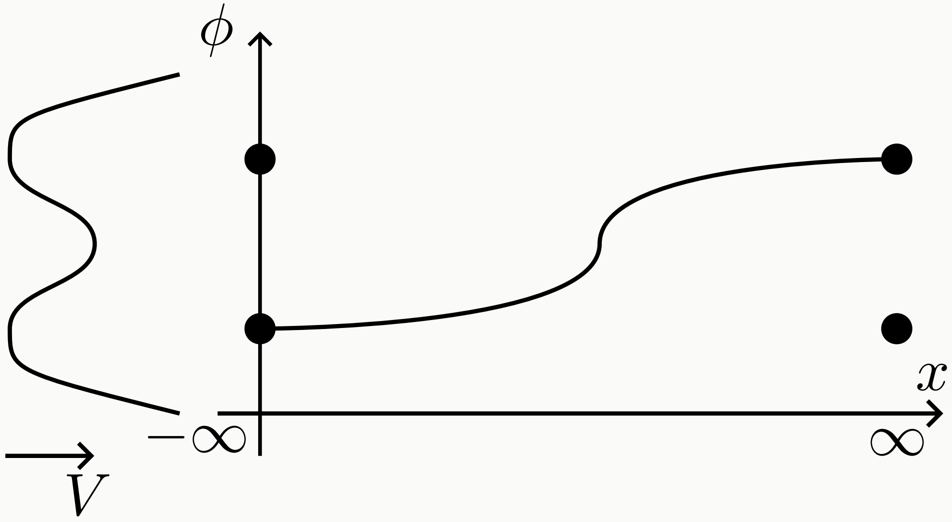
  where $\rho_e(g)$ is a representation matrix.
- Excitation $|e\rangle$ is classified by its **irreducible representation** of G.

Completely Broken Phase

- G: finite group (e.g. $\mathbb{Z}_2, S_3, \cdots$)
- Complete spontaneous symmetry breaking: $G \rightsquigarrow \{*\}$.
- $|G|$ degenerate vacua.
- $v \in G \Rightarrow$ vacuum $|v\rangle$.
  - Microscopic symmetry $g \in G$ acts as $g|v\rangle = |gv\rangle$.

Solitons in SSB phase

- Soliton $|s(v_1, v_2)\rangle$ interpolates between v_1 and v_2.
- $g|s(v_1, v_2)\rangle = |s(gv_1, gv_2)\rangle$.
- G-action generates **$|G|$-fold multiplet**.



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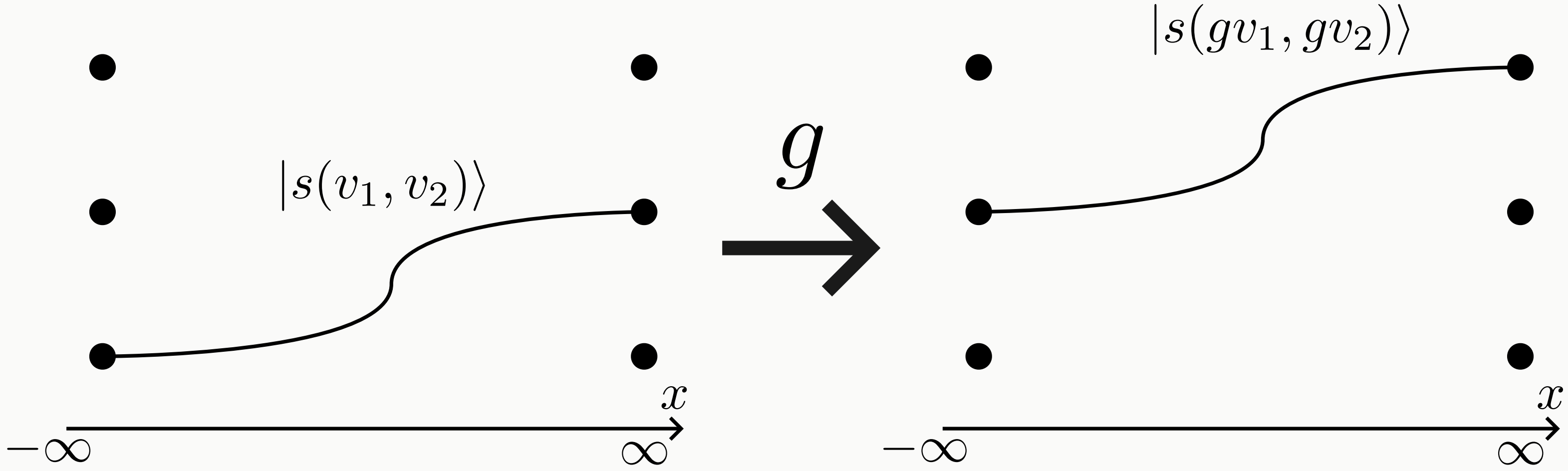

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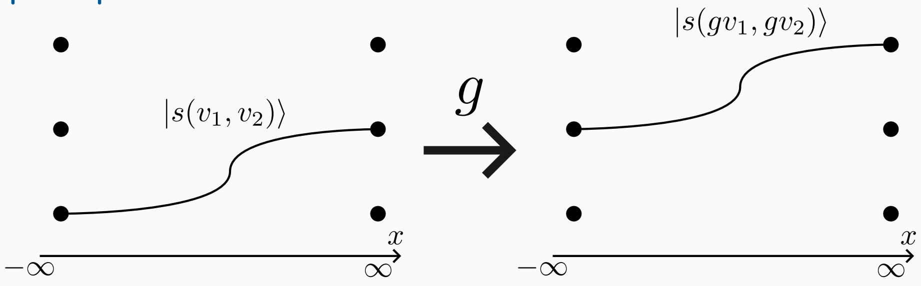
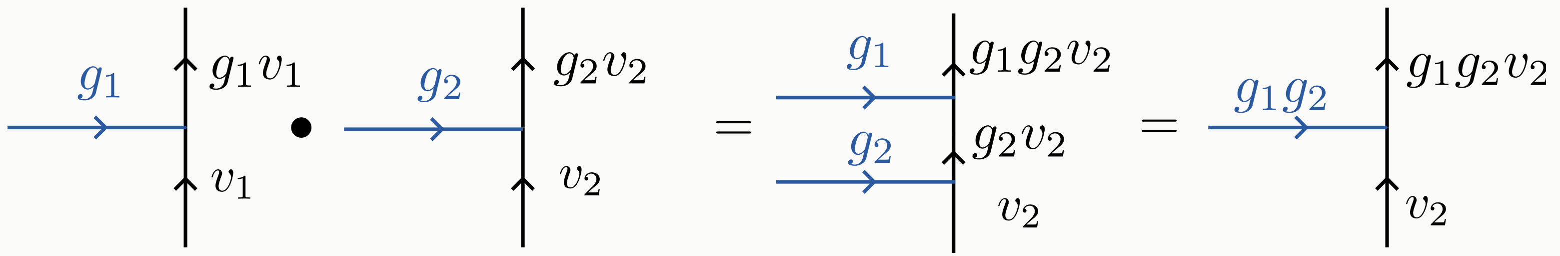


> Symmetry is "non-linearly represented".
> A better understanding is that states in the phase are in (linear) representations of a **groupoid algebra**.

Action Groupoid

- "Groupoid" is something like a group, but with a partial multiplication.
- We consider the action groupoid $X /\!\!/ G$ for a set X with an G-action.
  - $v \in X$ is a vacua (at the left ∞, say).
  - As a set, $X /\!\!/ G = X \times G$, and denote an element as $\xleftarrow{g} v$.
  - Multiplication : $(\xleftarrow{g_1} v_1) \cdot (\xleftarrow{g_2} v_2) = (\xleftarrow{g_1 g_2} v_2)$ if $v_1 = g_2 v_2$, otherwise undefined.
- $* /\!\!/ G = G$.

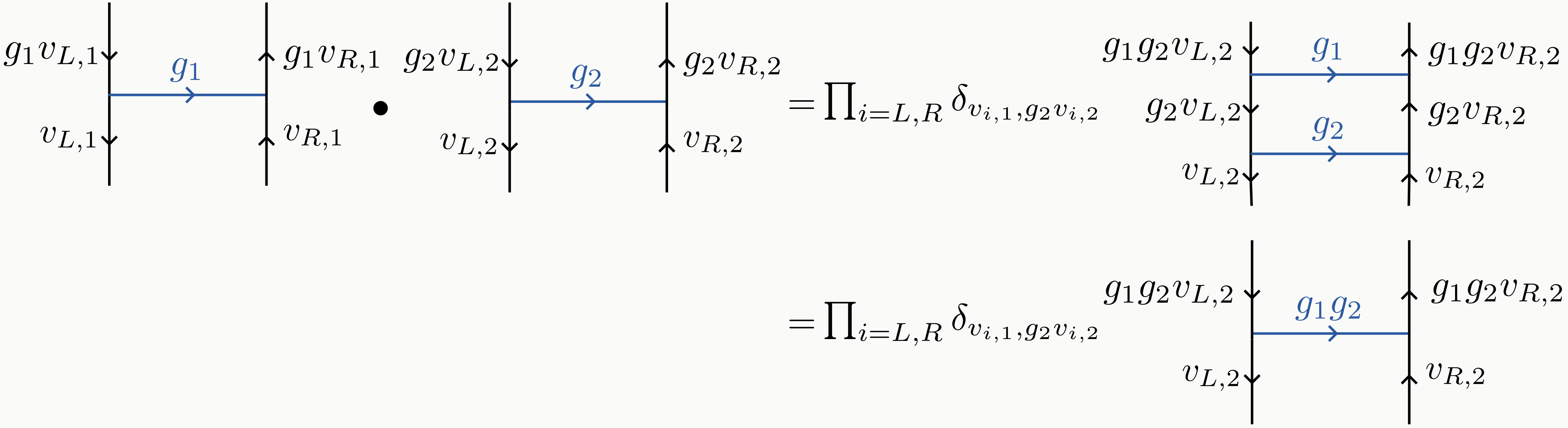


Groupoid Algebra

- The groupoid algebra $\mathbb{C}[X /\!/ G]$: a vector space spanned by $\xleftarrow{g} v$:
  - i.e. $\mathbb{C}[X /\!/ G] \ni \sum_{g,v} c_{g,v} [\xleftarrow{g} v]$
  - Multiplication: $[\xleftarrow{g_1} v_1] \cdot [\xleftarrow{g_2} v_2] = \delta_{v_1, g_2 v_2} [\xleftarrow{g_1 g_2} v_2]$.
- **Representation of groupoid algebra** (R, ρ) is an algebra hom $\rho : \mathbb{C}[X /\!/ G] \to \mathrm{GL}(R)$ for a Hilbert space R.
- For $X = *$, (R, ρ) is a usual group representation.
- $C(X) \rtimes \mathbb{C}[G]$ (Ojima 2002)

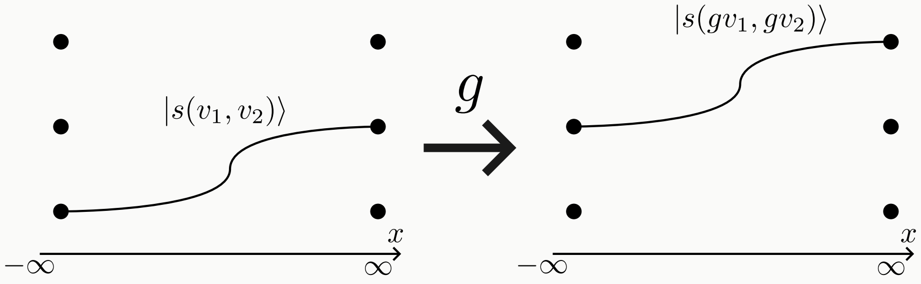
Groupoid for SSB

- For the SSB phase where G is completely broken, the relevant groupoid is $G^2 /\!\!/ G$, where the action is $(v_1, v_2)g = (v_1 g, g^{-1} v_2)$.
- $(v_1, v_2) \in G^2$: vacua at left and right infinity.
- Multiplication in $\mathbb{C}[G^2 /\!\!/ G]$:

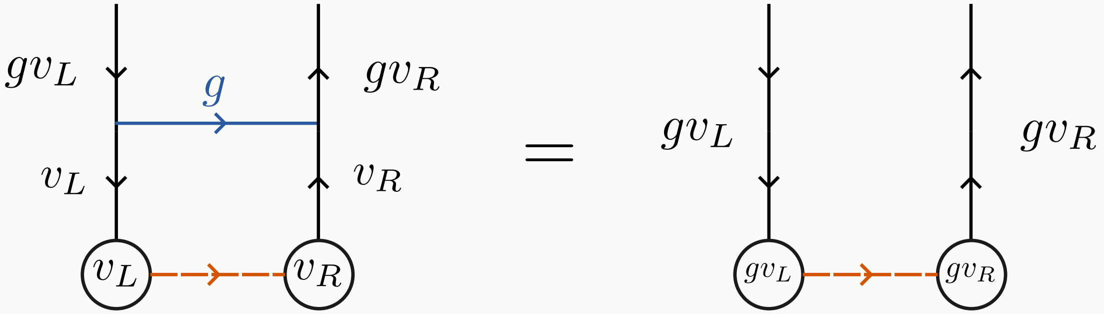


Groupoid Algebra Representation

- $[\overset{g}{\leftarrow}(v_L, v_R)]|s(v'_L, v'_R)\rangle = \delta_{v_L,v'_L}\delta_{v_R,v'_R}|s(gv_L, gv_R)\rangle$



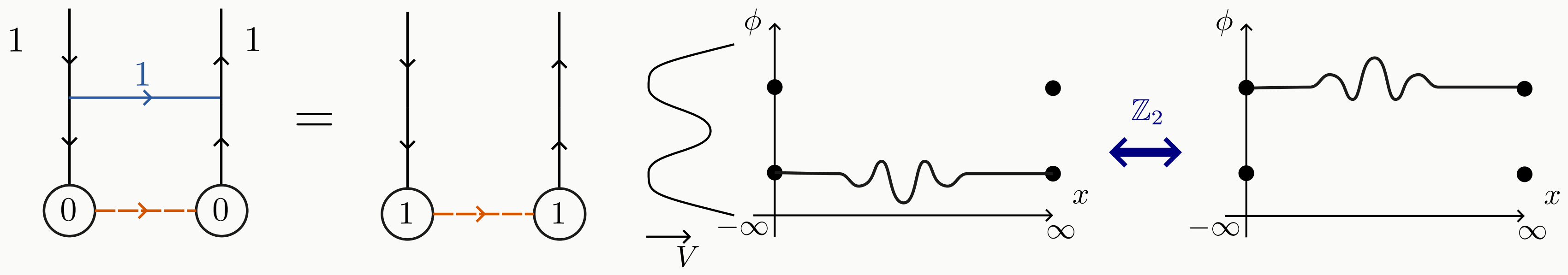
- Diagram:



Groupoid Algebra Representation for $\mathbb{Z}_2$ SSB

- Irreps of $\mathbb{C}[\mathbb{Z}_2^2 /\!/ \mathbb{Z}_2]$: R_0, R_1.

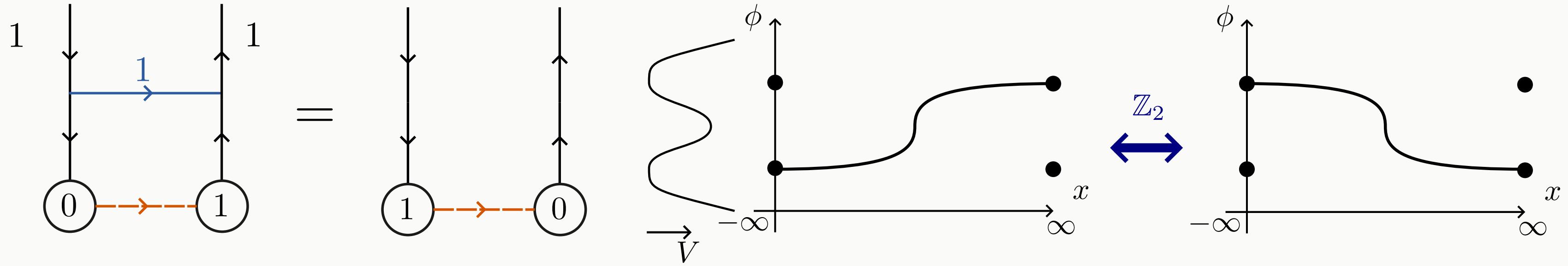
- R_0 : "particle" sector : $(v_L, v_R) = (0,0), (1,1)$



Groupoid Algebra Representation for $\mathbb{Z}_2$ SSB

- Irreps of $\mathbb{C}[\mathbb{Z}_2^2 /\!\!/ \mathbb{Z}_2]$: R_0, R_1.

  - R_1 : "soliton/antisoliton" sector : $(v_L, v_R) = (0,1), (1,0)$.



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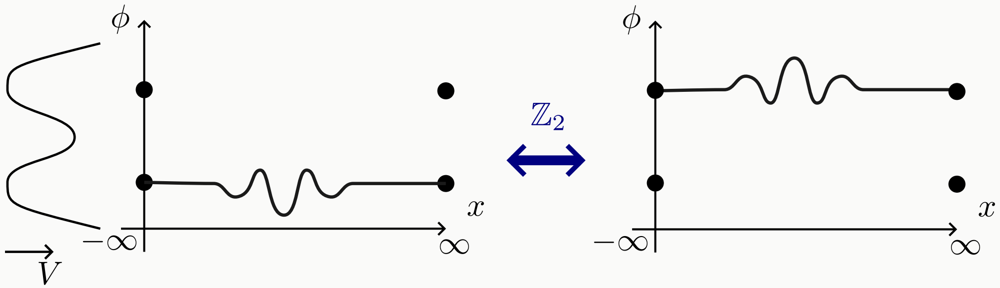

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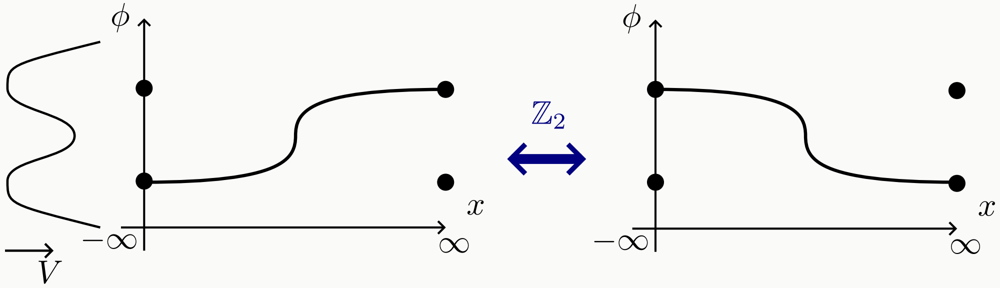


Tensor Product and Scattering Selection Rule

- $G = \mathbb{Z}_2$ SSB case
- "Tensor product" of representations: R_i "$\otimes$" $R_j = R_{i+j \mod 2}$.
- E.g. $R_1 \otimes R_1 = R_0$:
  - (soliton)$\otimes$ (antisoliton) $\longrightarrow$ (particle).
  - (soliton)$\otimes$ (antisoliton) $\not\rightarrow$ ((anti)soliton).

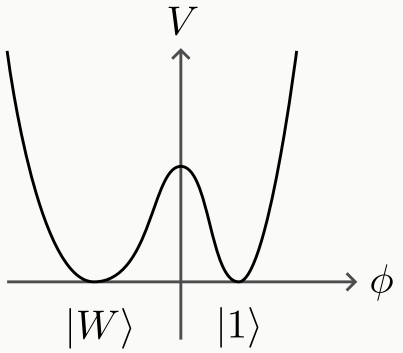
> Tensor product dictates the **selection rule**!

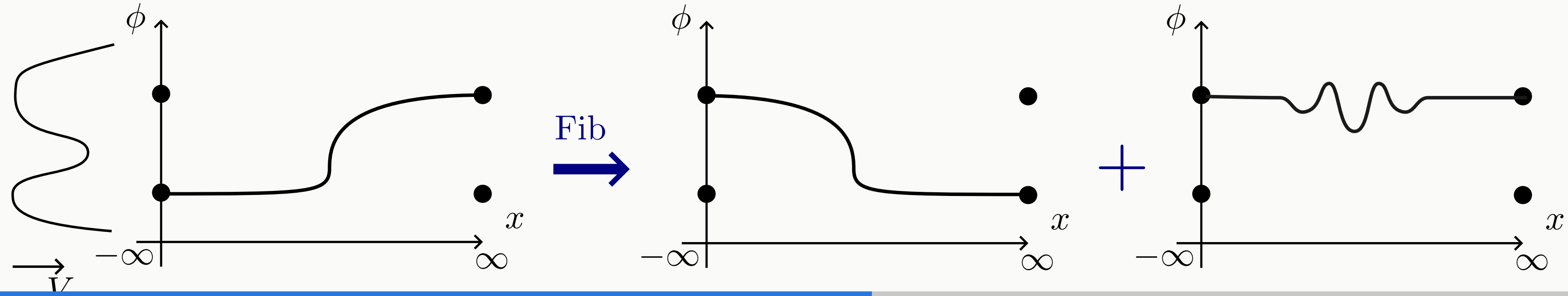
Rep. Theory for Solitons w/ Non-inv. Sym.

(Cordova, Holfester, and Ohmori 2024)

- Generalizes to **non-invertible symmetry**
- $\rightsquigarrow$ "**strip algebra**" (Kitaev and Kong 2012; Johnson-Freyd and Reutter WIP)
- E.g. Deformed tricritical Ising model (Zamolodchikov 1990)
  - SSB'en "Fibonacci" sym. (Chang et al. 2019)
  - a multiplet involving **both** particles and solitons!

    (Córdova, García-Sepúlveda, and Holfester 2024)





General Theory

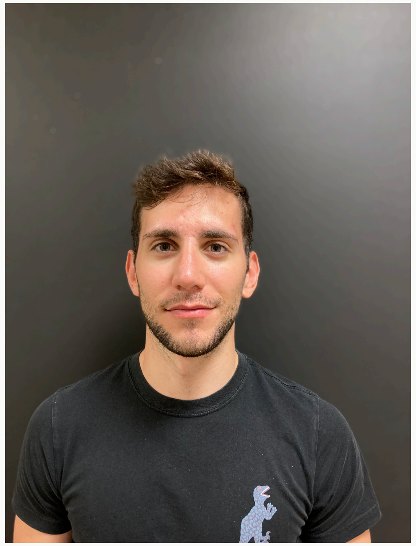
symbol	math	physics
$\mathcal{C}$	fusion category	UV symmetry/ Topological lines
$\mathcal{M}$	module category	IR gapped phase
$\mathbf{Str}_{\mathcal{C}}(\mathcal{M})$	weak Hopf alg.	Symmetry action on $\mathbb{R}^2$
$\mathrm{Rep}_{\mathcal{C}}(\mathcal{M}) \cong \mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(\mathcal{M})$	Dual Category	**Excitation multiplets**

> While $\mathbf{Str}_{\mathcal{C}}(\mathcal{M})$ is often cumbersome, one can often directly compute the representation category $\mathrm{Rep}_{\mathcal{M}}(\mathcal{C})$. E.g. full SSB: $\mathrm{Rep}_{\mathcal{C}}(\mathcal{C}) \cong \mathcal{C}$.

Representation Theory for Gauge Theories

Based on (Gagliano, Grigoletto, and Ohmori 2025, WIP)

Finn Gagliano



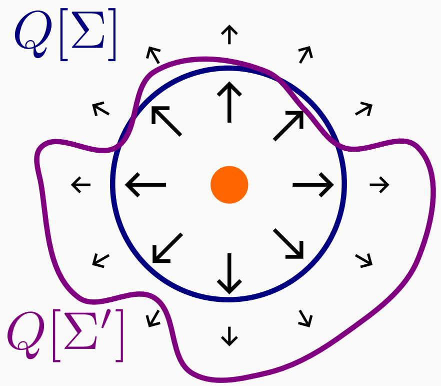
Andrea Grigoletto

Higher Form Symmetry (Gaiotto et al. 2015)

Conventional : $Q = \int_{\text{space}} d^d x \ \rho(x)$

p-form sym. : $Q[\Sigma] = \int_\Sigma d^{d-p} x \ \rho(x)$

- Σ : $d - p$ dim. surface in spacetime
- **Conservation law**: $Q[\Sigma] = Q[\Sigma']$ if deformable to each other
  - In particular, $\frac{d}{dt} Q[\Sigma(t)] = 0$
- Example: **Gauss's law** in electromagnetism : $U(1)^{(p=1)}$
  - Transformation law : $A \mapsto A + \Lambda \ (d\Lambda = 0)$
  - **photon** : Nambu-Goldstone mode!



Higher Strip Algebra

- 1+1d: $C((G/H)^2) \rtimes \mathbb{C}[G]$
  - Rep. category $\cong \mathrm{Vect}_{G/H} \otimes_\omega \mathrm{Rep}(H)$
- d+1-dim: **Higher strip algebra** (d-algebra)
  - "Higher vector space" ($\cong (d-1)+1$-dim TQFT $\mathcal{T}$)
  - "multiplication" $\mathcal{T} \boxtimes \mathcal{T} \rightsquigarrow \mathcal{T}$.
  - Representation higher category $\cong d\mathrm{Vect}_{G/H} \otimes_\omega d\mathrm{Rep}(H)$
- $\mathrm{Rep}_{\mathcal{C}}(\mathcal{M}) \cong \mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(\mathcal{M})$ (Assumed)

Higher Form Symmetry in Gauge theory

- pure YM with non-ab. gauge group $G \Rightarrow Z(G)^{(1)}$ **form symmetry**
  - e.g. $G_{\text{gauge}} = SU(N) : \mathbb{Z}_N^{(1)}$ one-form symmetry
  - $W_{\square}$ cannot be screened by gauge fields.

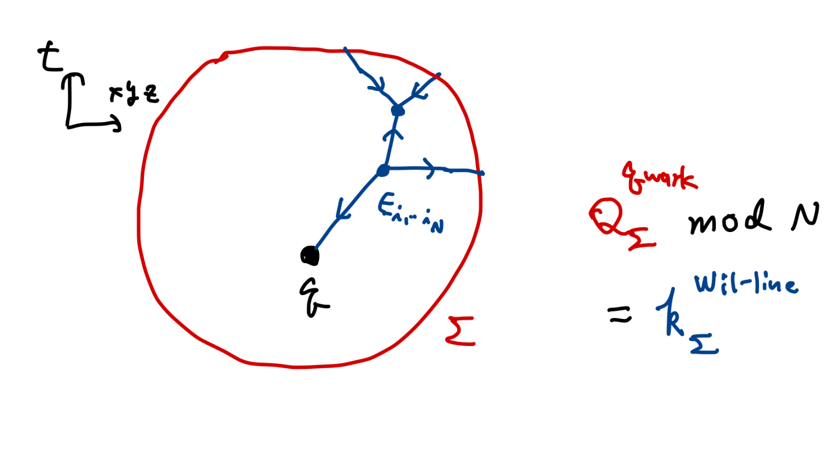
confinment (or not) $\Leftrightarrow \mathbb{Z}_N^{(1)}$ is preserved (or SSB′en).

Symmetry in single flavor QCD

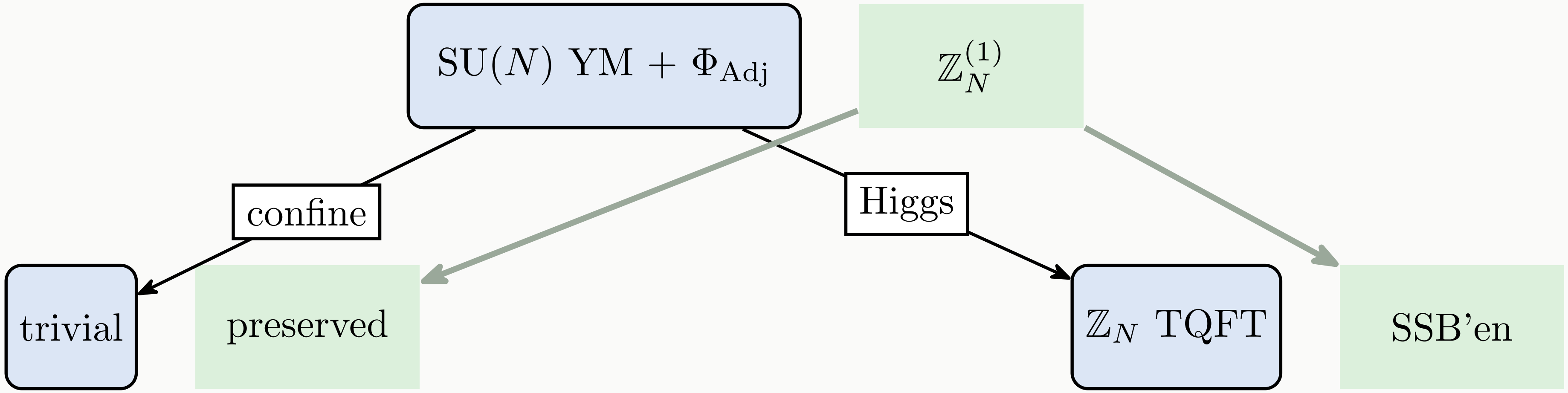
- $G_{\text{gauge}} = \mathrm{SU}(N)$ + a (scalar) quark q in $\Box$
- $U(1)_q : q \mapsto e^{i\theta} q$: quark rotation
- $(\mathbb{Z}_N)_q = Z(G_{\text{gauge}}) \subset G_{\text{gauge}}$: gauged!
- $G_{\text{UV}} = U(1)_B = U(1)_q/\mathbb{Z}_N$: **Baryon number**
  - A local (gauge invariant) operator has **integer** Baryon charge.
  - $q_{i_1} \partial q_{i_2} \cdots \partial^N q_{i_N} \epsilon^{i_1, \cdots, i_N}$

Symmetry Matching for Heavy Quark

- m_q: quark mass
  - $\mu \leq m_q$: $\mathbb{Z}_N^{(1)}$: **emergent** one-form symmetry
- Symmetry matching homomorphism: $U(1)_B \xrightarrow{\beta_N} \mathbb{Z}_N^{(1)}$
  - **Non-trivial!** (Bockstein map) c.f. (Seiberg and Seifnashri 2025)

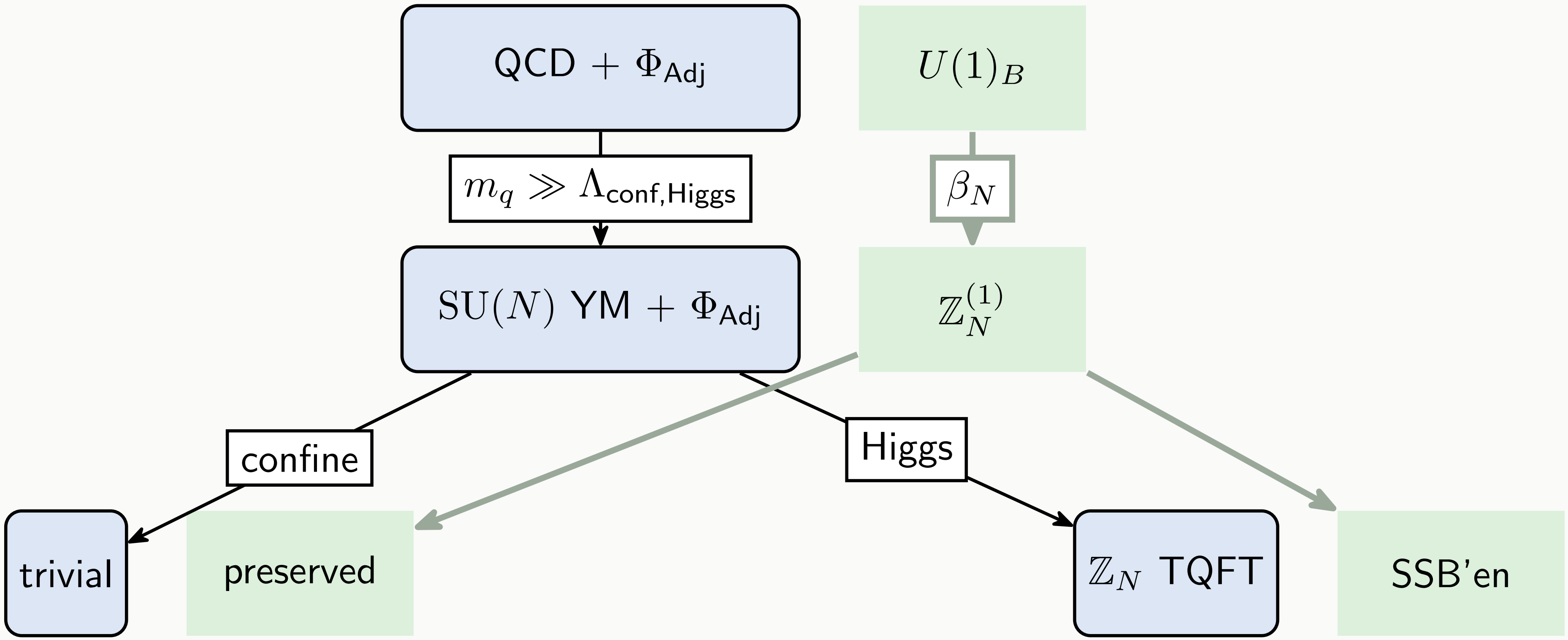


Higher Representation Theory



$3\mathrm{Rep}(\mathbb{Z}_N^{(1)}) \ni$ **Confining String** $3\mathrm{Vec}_{\mathbb{Z}_N^{(1)}} \ni$ **Center Vortex**

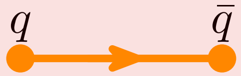
Higher Representation Theory



$\mathrm{3Rep}(\mathbb{Z}_N^{(1)}) \ni$ Confining String

$\downarrow \beta_N^*$

$\mathrm{3Rep}(U(1)_B) \ni$ Baryon,



$\mathrm{3Vec}_{\mathbb{Z}_N^{(1)}} \ni$ Center Vortex

$\downarrow \beta_N^*$

$\mathrm{3Rep}(U(1)_q) \boxtimes_\omega \mathrm{3Vec}_{\mathbb{Z}_N^{(1)}} \ni$ **Quark**, Vortex

Adjoint Higgs Phase

- $G_{\text{UV}} = U(1)_B \xrightarrow{\mu<m_q} \mathbb{Z}_N^{(1)} \xrightarrow{\mu<v_\Phi} *$
- "Exact sequence": $U(1)_q \to U(1)_B \to \mathbb{Z}_N^{(1)}$
  - "$\mathbb{Z}_N^{(1)} = U(1)_B / U(1)_q$"
  - "break **up**" : $U(1)_B \rightsquigarrow U(1)_q$
- **quark** : $\rho \in \text{Rep}(H = U(1)_q)$
- **center vortex**: $\sigma \in \text{Rep}(G/H = \mathbb{Z}_N^{(1)})$.
- Similar analysis for superconductor phase in abelian gauge theory.

Summary

Message

Represntation theory for excitations **depends on the phase**

- Discret symmetery SSB phase $\Rightarrow$ groupoid algebra representation
- 1+1d SSB phase $\Rightarrow$ **solitons**
- (scalar) QCD
  - One-form symmetry pres / SSB
    $\Rightarrow$ **Representation theory** does not/ does contain quark

Prospects

Our work: **bosonic** theory in **gapped** and **zero-temperature** phases.

- Generalizations
  - Thermal Phase
    - Full QCD phase?
  - Gapless phases
    - Full gauge charge in Coulomb phase?
    - Quantum Gravity?

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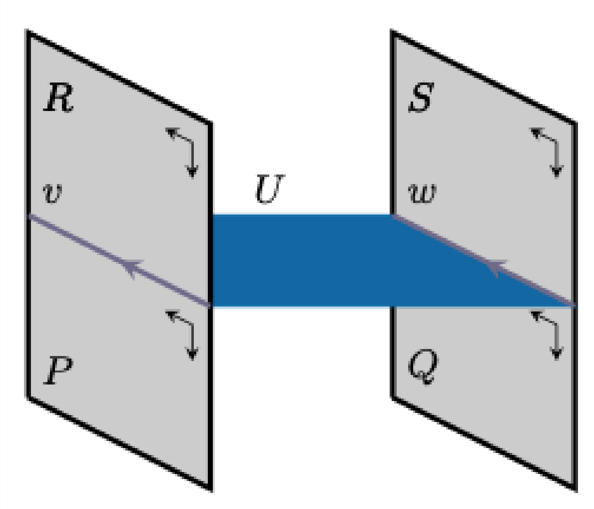
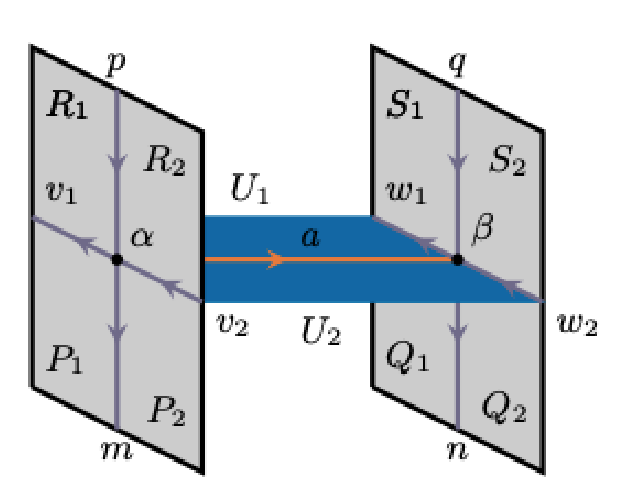
Thank you !!

Higher Strip Algebra

- In $d+1$ QFT, higher strip algebra:
  - an weak Hopf algebra object in $d\mathrm{Vect}$.
  - Rep. theory $\cong \mathrm{Vect}_{G/H} \otimes_\omega \mathrm{Rep}(H)$



object ($d=2$)



1-morphism ($d=2$)

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